Math 10A
Worksheet, Final Review; Thursday, 8/9/2018
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1 Review Topics

1.1 Differentiation

- Domain/Range of functions
- Function transformations (Draw $f(2x+3)$)
- Limits
  - Infinite limits
  - L'Hopital's Rule
- Tangent Lines
  - Tangents to inverse functions
- Derivatives
  - Product Rule, Quotient Rule
  - Chain Rule
  - Implicit Differentiation
- Graphing Functions
  - Local extrema
  - Global extrema
  - Critical points
  - Concavity
  - Second Derivative Test
- Optimization
- Related Rates
- Taylor Series
- Newton's Method

1.2 Integration

- Antiderivatives
  - Fundamental Theorem of Calculus I and II
- Substitution Rule
- Integration by Parts
- Symmetry
- Numerical integration
  - Left/Right/Midpoint/Trapezoid/Simpson's Rule
  - Error Bounds
- Improper Integrals
  - Convergence Test
- Partial Fractions

1.3 Differential Equations

- Recurrence Relations
  - Going both forward and backward
  - Verifying solutions
- Identifying the adjectives (linear, homogeneous, etc.)
- Integrating Factors
- Separable Equations
- Second order differential equations
  - Going forward and backward
- IVPs/BVPs
- Slope fields
  - Euler's Method
  - Logistic Growth
- Linear systems of differential equations

1.4 Matrices

- Multiplying matrices, vectors
- Determinants
  - Number of solutions and how it depends on the determinant
- Gaussian Elimination
  - Finding Inverses
  - Solving matrix-vector equations
- Eigenvalues/eigenvectors
- Linear Regression
  - Least Squares Error
  - Finding line of best fit

2 Recurrence Relations and 2nd order Differential Equations

1. **TRUE** False It is possible for an IVP to have a unique solution.
2. **TRUE** False It is possible for a BVP to have a unique solution.
3. True **FALSE** It is possible for an IVP to have infinitely many solutions.
4. **TRUE** False It is possible for a BVP to have infinitely many solutions.
5. Solve the recursion equation $a_n = 2a_{n-2} - a_{n-1}$ with the initial conditions $a_0 = 0, a_1 = 3$.

Solution: The characteristic equation is $\lambda^2 = 2 - \lambda$ or $\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0$ so $\lambda = 1, -2$ are the roots. Therefore, the general solution is $c_1(1)^n + c_2(-2)^n$ or $c_1 + c_2(-2)^n$. The initial conditions give $c_1 + c_2 = 0$ and $c_1 - 2c_2 = 3$. Adding twice the first to the second gives $3c_1 = 3$ so $c_1 = 1, c_2 = -1$. Therefore, the solution is $1 - (-2)^n$.

6. Verify that $y_1(t) = t$ and $y_2(t) = t^3$ are solutions to the differential equation $t^2y''(t) - 3ty'(t) + 3y(t) = 0$. Find the solution to the differential equation with $y(1) = 2$ and $y'(1) = 4$ (hint: what kind of differential equation is this?).

Solution: We plug in t and t^3 and show that we get an equality. Since this is a homogeneous linear polynomial, linear combinations of solutions are also solutions so $c_1t+c_2t^3$ is the general solution. Plugging in the initial condition gives $c_1+c_2=2$ and $c_1+3c_2=4$ so $c_1=c_2=1$. Therefore, the solution is $y(t)=t+t^3$.

7. Find all solutions to the BVP $y''+2y'+5y=0$ with $y(0)=0$ and $y(\pi)=0$.

Solution: The characteristic equation is $\lambda^2+2\lambda+5=0$ so the roots are $\lambda=-1\pm 2i$. Therefore, the general solution is $c_1e^{-t}\sin(2t)+c_2e^{-t}\cos(2t)$. Now plugging in the initial conditions give $c_2=0$ and $c_2e^{-\pi}=0$ or $c_2=0$. Therefore, the solution is $y(t)=c_1e^{-t}\sin(2t)$ and there are infinitely many solutions.

8. Find all solutions to the BVP $y''-5y'+6y=0$ with $y(0)=2$ and $y(1)=e^2+e^3$.

Solution: The characteristic equation is $\lambda^2-5\lambda+6=(\lambda-2)(\lambda-3)=0$. So the roots are $\lambda=2,3$ and the general solution is $y(t)=c_1e^{2t}+c_2e^{3t}$. Plugging in the initial conditions give $c_1+c_2=2, c_1e^2+c_2e^3=e^2+e^3$ so $c_1=c_2=1$. Therefore, the unique solution is $y(t)=e^{2t}+e^{3t}$.

9. Solve the initial value problem $3y''+18y'+27y=0$ with $y(0)=0, y'(0)=1$.

Solution: We guess the solution is of the form $y=e^{rt}$. Plugging this in gives $3r^2e^{rt}+18re^{rt}+27e^{rt}=23e^{rt}(r^2+3r+9)=0$ and hence $(r+3)^2=0$ so $r=-3$ is a double root. Therefore, the general solution is of the form $y=c_1e^{-3t}+c_2te^{-3t}$. Plugging in our initial conditions gives $y(0)=0$ or $0=c_1e^0+c_2(0)(e^0)=c_1$ and $y(t)=c_2te^{-3t}$ and $y'(t)=c_2t(-3e^{-3t})+c_2e^{-3t}$ and plugging in $y'(0)=1$ gives $c_2(0)+c_2(1)=c_2=1$ so the solution is $y(t)=te^{-3t}$.

10. Solve the initial value problem given by $2y''=3y'-y$ and $y(0)=0$ and $y'(0)=1$.

Solution: We bring all the y's to one side and get $2y''-3y'+y=0$ and our characteristic equation is $2r^2-3r+1=(2r-1)(r-1)=0$ so $r=1/2,1$. So the solution is of the form $y(t)=c_1e^{t/2}+c_2e^t$. Plugging in the initial conditions gives $y(0)=c_1+c_2=0$ and $y'(0)=c_1/2+c_2=1$ which solving gives $c_2=2$ and $c_1=-2$. Therefore the solution is $y(t)=-2e^{t/2}+2e^t$.

11. Find a second order differential equation IVP that has te^t as a solution.

Solution: Since te^t is a solution, the t in front tells us that there is a double root and the e^t tells us that $\lambda = 1$ is a root. Therefore, the roots are $\lambda = 1, 1$. So the characteristic equation is $(\lambda - 1)(\lambda - 1) = \lambda^2 - 2\lambda + 1 = 0$. So the differential equations is $y'' - 2y' + y = 0$. The conditions are $y(0) = 0e^0 = 0, y'(0) = te^t + e^t|_{t=0} = 0e^0 + e^0 = 1$.

12. Find a second order differential equation BVP that has $e^{2t}\sin(t)$ as a solution.

Solution: Since we have sin, we know that there are complex roots so the roots are $\lambda = a \pm bi$. The a is the exponent of e^{2t} so $a = 2$ and b is in the sin or cos so $b = 1$. Therefore, $\lambda = 2 \pm i$ are the roots. So the characteristic equation is $(\lambda - (2 - i))(\lambda - (2 + i)) = 0$ or $\lambda^2 - 4\lambda + 5 = 0$. Therefore, the differential equation is $y'' - 4y' + 5y = 0$.

The initial conditions is a boundary value so we could take $y(0) = e^0 \sin(0) = 0$ and $y(1) = e^2 \sin(1)$.

13. Find the second order linear ODE such that $y(t) = te^{2t}$ is a solution to it.

Solution: Since te^{2t} is a solution, this tells us that 2 is a double root. Now in order to find the characteristic equation, we just multiply $(r - 2)^2 = r^2 - 4r + 4$. So, the ODE is $y'' - 4y' + 4y = 0$. The initial conditions are $y(0) = 0, y'(0) = 1$.

14. What is the largest value of $\alpha > 0$ such that any solution of $y'' + 4y' + \alpha y = 0$ does not oscillate (does not have any terms of $\sin, \cos$).

Solution: The characteristic equation is given by $r^2 + 4r + \alpha = 0$. The roots are $\frac{-4 \pm \sqrt{16 - 4\alpha}}{2}$ and this does not have any terms of $\sin, \cos$ whenever $16 - 4\alpha \geq 0$ or when $\alpha \leq 4$. Therefore, the largest value is $\alpha = 4$.

3 First Order Differential Equations

15. Solve the differential equations $(t^3 + t^2)y' = \frac{t^2 + 2t + 2}{2y}$ with the initial condition $y(1) = 1$.

Solution: This is first order but not linear so we use separable equations to get

$$2ydy = \frac{t^2+2t+2}{t^3+t^2}dt$$

We rewrite the right side using partial fractions ($t^3+t^2 = t^2(t+1)$) as $\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t+1}$. Solving for A, B, C gives $A=0, B=2, C=1$ and integrating gives

$$y^2 = \int 2ydy = \int \frac{t^2+2t+2}{t^3+t^2}dt = \int \frac{2}{t^2}+\frac{1}{t+1}dt = \frac{-2}{t}+\ln|t+1|+C.$$

Now we plug in our initial condition $y(1)=1$ to get $1=-2+\ln 2+C$ and $C=3-\ln 2$ so

$$y=\sqrt{-2/t+\ln|t+1|+3-\ln 2}.$$

16. Consider the differential equation $ty'+3y=5t^2$ with initial condition $y(1)=1$. Draw a slope field and then estimate $y(5)$ using a step size of $h=2$. Then solve for y explicitly and find the exact value of $y(5)$.

Solution: Moving things over, we get $y'=5t-3y/t=f(t,y)$. Our initial condition is the point $(t_0,y_0)=(1,1)$. Then our next point has $t_1=1+h=3$ and $y_1=y_0+hf(t_0,y_0)=1+2(5(1)-3(1/1))=5$. So we have the point $(3,5)$. Now to get the next point we have $t_2=t_1+h=5$ and $y_2=y_1+hf(t_1,y_1)=5+2(5(3)-3(5/3))=25$. So the next point is $(5,25)$ and $y(t)$ is about 25.

To find the exact solution, we need to solve for y. This is a first order linear equation so we use integrating factors. To do this, we write it as $y'+\frac{3}{t}y=5t$. We multiply by the integrating factor which is $e^{\int 3/tdt}=e^{3\ln t}=t^3$ to get $t^3y'+3t^2y=5t^4$. Integrating gives

$$t^3y=\int (t^3y)'dt=\int t^3y'+3t^2ydt=\int 5t^4dt=t^5+C.$$

Thus, we have that $y=t^2+\frac{C}{t^3}$. Plugging in the initial condition $y(1)=1$ gives $1=1+C$ so $C=0$ and $y=t^2$ is the solution so $y(5)=25$.

17. For more Euler's method practice, see the Discussion 29 Worksheet.

18. Find all solutions to $e^ty'=y^2+2y+1$.

Solution: This is a separable equation because we can write $y'=(y+1)^2e^{-t}$. This gives $\frac{dy}{(Y+1)^2}=e^{-t}dt$ so integrating gives $\frac{-1}{y+1}=-e^{-t}+C$ so $\frac{1}{y+1}=e^{-t}+C$. Therefore, solving gives $y=\frac{1}{e^{-t}+C}-1$. There is also a missing solution when we divided which was $y=-1$.

19. Find the solution of $y' + \frac{y}{x} = e^x/x$ with $y(1) = 0$.

Solution: The integrating factor is $e^{\int 1/x dx} = e^{\ln x} = x$ and multiplying through gives us $xy' + y = (xy)' = e^x$. Now integrating gives $xy = e^x + C$ and hence $y = \frac{e^x}{x} + \frac{C}{x}$. Now solving for the initial condition gives us $0 = \frac{e^1}{1} + \frac{C}{1} = e + C$. Hence $C = -e$ so $y = \frac{e^x}{x} - \frac{e}{x}$.

20. Find the solution to $r' = r^2/t$ with $r(1) = 1$.

Solution: Split it to get $\frac{dr}{r^2} = \frac{dt}{t}$ and now integrating gives $-1/r = \ln t + C$ or $r = \frac{-1}{\ln t + C}$. Now we plug in the initial condition of $r(1) = 1$ and since $\ln 1 = 0$, we have that $1 = \frac{-1}{0+C} = \frac{-1}{C}$. Hence $C = -1$ and so the solution is $r(t) = \frac{-1}{-1+\ln t}$.

21. Find the general solution to $y' + 2y/x = \sin(x)/x^2$.

Solution: This is linear so we want to use integrating factors. Let $I(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$. Multiplying by x^2, we get $x^2y' + 2xy = \sin(x)$. Now we integrate both sides to get $(x^2y) = -\cos(x) + C$ so $y = \frac{C - \cos(x)}{x^2}$.

22. Find the general solution of $y' = 2t \sec y$.

Solution: Separating by dividing by $\sec y$ gives us $dy/\sec(y) = \cos(y)dy = 2tdt$. Now integrating gives $\sin(y) = t^2 + C$ which is the general solution.

23. Find the general solution to $y' - 2y/x = 3x^3$.

Solution: This is linear so we want to use integrating factors. Let $I(x) = e^{\int \frac{-2}{x} dx} = e^{-2\ln x} = x^{-2}$. Multiplying by $I(x)$ gives us $x^{-2}y' - 2y/x^3 = 3x$. Integrating both sides gives us $I(x)y = x^{-2}y = 3x^2/2 + C$ so $y = 3x^4/2 + Cx^2$.

4 Matrices

24. True **FALSE** If A, B are square $n \times n$ matrices, then $AB = BA$.

25. True **FALSE** If A is a 2×2 matrix such that $A^2 = I_2$, then $A = I_2$.

26. Find the general solution to the systems of linear differential equations

$$\begin{cases} y_1'(t) = 2y_1(t) + y_2(t) \\ y_2'(t) = y_1(t) + 2y_2(t) \end{cases}$$

Solution: Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then letting $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, we have that $\vec{y}' = A\vec{y}$. The eigenvalues of A are given by $(2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3$ or $\lambda = 1, 3$. For $\lambda = 1$, the eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and for $\lambda = 3$, the eigenvector is given by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, the general solution is

$$\vec{y} = c_1 e^t \vec{v}_1 + c_2 e^{3t} \vec{v}_2 = \begin{pmatrix} c_1 e^t + c_2 e^{3t} \\ -c_1 e^t + c_2 e^{3t} \end{pmatrix}.$$

27. Find the general solution to the systems of linear differential equations

$$\begin{cases} y_1'(t) = y_1(t) + 4y_2(t) \\ y_2'(t) = 2y_1(t) + 3y_2(t) \end{cases}$$

Solution: Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Then letting $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, we have that $\vec{y}' = A\vec{y}$. The eigenvalues of A are given by $(1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$ or $\lambda = -1, 5$. For $\lambda = -1$, the eigenvector is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and for $\lambda = 5$, the eigenvector is given by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, the general solution is

$$\vec{y} = c_1 e^{-t} \vec{v}_1 + c_2 e^{5t} \vec{v}_2 = \begin{pmatrix} -2c_1 e^{-t} + c_2 e^{5t} \\ c_1 e^{-t} + c_2 e^{5t} \end{pmatrix}.$$

28. Find the general solution to the systems of linear differential equations

$$\begin{cases} y_1'(t) = -y_1(t) + 3y_2(t) \\ y_2'(t) = 2y_1(t) \end{cases}$$

Solution: Let $A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$. Then letting $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, we have that $\vec{y}' = A\vec{y}$. The eigenvalues of A are given by $(-1-\lambda)(0-\lambda) - 6 = \lambda^2 + \lambda - 6$ or $\lambda = 2, -3$. For $\lambda = -3$, the eigenvector is $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and for $\lambda = 2$, the eigenvector is given by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, the general solution is

$$\vec{y} = c_1 e^{2t}\vec{v}_1 + c_2 e^{-3t}\vec{v}_2 = \begin{pmatrix} -3c_1e^{2t} + c_2e^{-3t} \\ 2c_1e^{2t} + c_2e^{-3t} \end{pmatrix}.$$

29. Consider the following set of points: $\{(0,6), (1,3), (2,1), (3,0), (4,0)\}$. Find the line of best fit through these points and use it to estimate $y(0.5)$.

Solution: We can calculate the line $y = ax+b$ as $a = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \frac{-15}{10} = \frac{-3}{2}$ and $b = \bar{y} - a\bar{x} = 2 - 2(-3/2) = 5$. So the line of best fit is $y = 5 - 3/2x$. We have $y(0.5) \approx 5 - 3/2(1/2) = 5 - 3/4 = 4.25$.

30. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$. Find A^{-1}.

Solution:

$$A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

31. Let $A = \begin{pmatrix} 3 & 4 & -1 \\ 4 & 2 & 1 \\ -2 & -3 & 1 \end{pmatrix}$. Find A^{-1}.

Solution: Use Gaussian elimination to get $A^{-1} = \begin{pmatrix} -5 & 1 & -6 \\ 6 & -1 & 7 \\ 8 & -1 & 10 \end{pmatrix}$.

32. Let A be the same as the previous problem. Solve $A\vec{x} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ (hint: use the previous problem to do this quickly).

Solution: The solution is $\vec{x} = A^{-1}\begin{pmatrix}1\\4\\-1\end{pmatrix}$ and we calculated A^{-1} in the previous problem to get $\vec{x} = \begin{pmatrix}5\\-5\\-6\end{pmatrix}$.

33. Let $\vec{v}_1 = \begin{pmatrix}3\\-2\\2\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}-1\\-1\\0\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}-1\\1\\-2\end{pmatrix}$ and suppose that A is a 3×3 matrix such that $A\vec{v}_1 = 4\vec{v}_1, A\vec{v}_2 = \vec{0}, A\vec{v}_3 = -\vec{v}_3$. What are the eigenvalues and eigenvectors of A? What is the general solution to $\vec{y}'(t) = A\vec{y}$ with $\vec{y} = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$?

Solution: The eigenvalues are $4, 0, -1$ with eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ respectively. The general solution is $e^{4t}\vec{v}_1 + e^{0t}\vec{v}_2 + e^{-t}\vec{v}_3$.

34. Let A be a 2×2 matrix and suppose that $\vec{y} = \begin{pmatrix}3e^{2t}+4e^{4t}\\e^{4t}-e^{2t}\end{pmatrix}$ is a solution to $\vec{y}' = A\vec{y}$. What are the eigenvalues and eigenvectors of A? What is $A\begin{pmatrix}3\\-1\end{pmatrix}, A\begin{pmatrix}4\\1\end{pmatrix}$ and $A\begin{pmatrix}7\\0\end{pmatrix}$?

Solution: We write the solution as $e^{2t}\begin{pmatrix}3\\-1\end{pmatrix} + e^{4t}\begin{pmatrix}4\\1\end{pmatrix}$. Therefore, one eigenvalue is 2 with eigenvector $\begin{pmatrix}3\\-1\end{pmatrix}$ and the other is 4 with eigenvector $\begin{pmatrix}4\\1\end{pmatrix}$. Then $A\begin{pmatrix}3\\-1\end{pmatrix} = 2\begin{pmatrix}3\\-1\end{pmatrix}$ and $A\begin{pmatrix}4\\1\end{pmatrix} = 4\begin{pmatrix}4\\1\end{pmatrix}$ and $A\begin{pmatrix}7\\0\end{pmatrix}$ is the sum.

35. Find the line of best fit through the points $\{(0,2),(1,3),(2,1)\}$.

Solution: We calculate $\bar{x} = 1, \bar{y} = 2$ and the line $y = ax+b$ is $a = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2} = \frac{-1}{2}$ and $b = \bar{y} - a\bar{x} = 2 - 1(-1/2) = 2.5$. So the line of best fit is $y = 2.5 - x/2$.

36. Write the differential equation $y'' + 5y' + 6y = 0$ as a systems of differential equations with $y_1(t) = y(t), y_2(t) = y'(t)$ and solve with $y(0) = 2, y'(0) = -5$.

Solution: We can write it as $y_1'(t) = y_2(t)$ and $y_2'(t) = -6y_1(t) - 5y_2(t)$ so it is represented in the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$. Solving gives us that $y_1(t) = y(t) = e^{-2t}+e^{-3t}$.